

Fragility distributions and their approximations

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Abstract

Given a sequence of n identically distributed random variables with common distribution F , the *fragility distribution of order m* , represented by $FD_{n,m}$, is the limit conditional distribution of the number of exceedances given there are at least m exceedances, as the threshold tends to the right end point of F . In this paper we are concerned with the existence of $FD_{n,m}$ and its asymptotic behaviour when n becomes large. For a stationary sequence with its exceedance process converging to a compound Poisson process, we derive an explicit formula for calculating $\lim_{n \rightarrow \infty} FD_{n,m}$. We also establish Stein's method for estimating the errors involved in fragility distribution approximations.

Key words and phrases: Exceedances, fragility distribution, compound Poisson approximation, Stein's method, Stein's factors.

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1 Introduction

The number of earthquake related claims made to an insurance company is typically zero. However, given the event that at least one claim was made, it is highly likely that multiple claims were lodged. A question worth considering then is, given at least one earthquake related claim occurred, what is the distribution of the total number of claims? This idea extends to the more general question: if there are at least m extreme events, what is the distribution of the number of extreme events? Aspects of this idea have been formalised in terms of the fragility index of order m , first introduced by Geluk et al. (2007). Given a stationary sequence of random variables X_1, \dots, X_n with common distribution function F_X , define the number of exceedances above the threshold s as:

$$N_{s,n} := \sum_{i=1}^n \mathbf{1}_{(s,\infty)}(X_i).$$

The extended fragility index of order m , denoted by $FI_n(m)$, is the asymptotic expected number of exceedances given that there are at least m exceedances:

$$FI_n(m) := \lim_{s \nearrow} \mathbb{E}(N_{s,n} | N_{s,n} \geq m),$$

where $s \nearrow$ is interpreted as “when s approaches the right end point $x_F := \sup\{t : F_X(t) < 1\}$ of F from below”.

It is well-known in statistics that expectations do not carry sufficient information for statistical inferences. In this paper, we will instead consider the fragility distribution of order m defined as

$$FD_{n,m} := \lim_{s \nearrow} \mathcal{L}(N_{s,n} | N_{s,n} \geq m).$$

In the case where X_1, \dots, X_n are independent and identically distributed (i.i.d.) with $\lim_{s \nearrow} (1 - F_X(s)) = 0$, it is simple to show $FD_{n,m}(\{m\}) = 1$ for

all $m \leq n$. In this case $FD_{n,m}$ exists for all $m \leq n$. In general though, given that $FD_{n,m}$ exists for some natural numbers m , does this imply the existence of $FD_{n,m}$ for other natural numbers? This question will be addressed in section 2.

Ultimately, it is the dependence structure that characterises the properties of $FD_{n,m}$. In section 3, we explore $FD_{n,m}$ for a stationary sequence as n tends towards infinity. Hsing et al. (1998) showed that under a mixing condition, if the exceedance point processes converge to a limit distribution, then the limit is of a compound Poisson type. Using this result, we establish a relation between the limiting compound Poisson distribution and fragility distributions, and give an explicit formula for calculating $\lim_{n \rightarrow \infty} FD_{n,m}$ for all $m \geq 1$.

In applications, we often face a fixed n and hence it is of interest to know the errors involved when approximations are used to replace the actual fragility distribution. In section 4 we focus on estimating errors of a conditional compound Poisson approximation using Stein's method. However, similar to compound Poisson approximation, Stein's constants for conditional compound Poisson approximation are generally crude and are of little value unless specific conditions are satisfied. This leads us to investigate Stein's factors for conditional compound Poisson approximation when the compounding distribution satisfies a certain condition and conditional negative binomial approximation. Finally, examples are provided to show that the errors are typically small in applications when these approximations are used to replace the fragility distributions.

2 Existence of fragility distributions

We consider the relationship between fragility distributions of different orders. By formulating the fragility distribution in the following manner the relationship between $FD_{n,m}$ and $FD_{n,m+1}$ becomes clearer. For all $A \subset$

$\{m, m+1, \dots, n\}$,

$$FD_{n,m}(A) = \lim_{s \nearrow} \frac{\mathbb{P}(N_{s,n} \in A)}{\mathbb{P}(N_{s,n} \geq m)} = \lim_{s \nearrow} \frac{\frac{\mathbb{P}(N_{s,n} \in A)}{\mathbb{P}(N_{s,n} \geq m+1)}}{1 + \frac{\mathbb{P}(N_{s,n}=m)}{\mathbb{P}(N_{s,n} \geq m+1)}}. \quad (2.1)$$

Notice that the numerator in (2.1) yields $FD_{n,m+1}$ if $\lim_{s \nearrow} \frac{\mathbb{P}(N_{s,n} \in A)}{\mathbb{P}(N_{s,n} \geq m+1)}$ exists. From this formulation we can see that if for all A , two of $FD_{n,m}(A)$, $FD_{n,m+1}(A)$ and $\lim_{s \nearrow} \frac{\mathbb{P}(N_{s,n}=m)}{\mathbb{P}(N_{s,n} \geq m+1)}$ exist, then the existence of the third is ensured. Hence, whether the existence of $FD_{n,m}$ implies the existence of $FD_{n,m+1}$ or vice versa depends on the existence of $\lim_{s \nearrow} \frac{\mathbb{P}(N_{s,n}=m)}{\mathbb{P}(N_{s,n} \geq m+1)}$.

The following counterexample shows that the existence of $FD_{n,m}$ does not guarantee the existence of $FD_{n,m+1}$. Neither does the existence of $FD_{n,m+1}$ guarantee the existence of $FD_{n,m}$.

To start, we define a density function on $[0, 1]$ as

$$g_1(y) = \begin{cases} 2 & y \in [1 - \frac{1}{2^k}, 1 - \frac{3}{2^{k+2}}] \text{ for } k \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $G_1(y)$ denote the distribution function of g_1 . By considering the two sequences $y_k = 1 - \frac{1}{2^k}$ and $y_k = 1 - \frac{3}{2^{k+1}}$, it can be shown that

$$\lim_{y \rightarrow 1} \frac{1 - G_1(y)}{1 - y} \quad (2.2)$$

does not exist.

Now we present an example where $FD_{n,2}$ exists, but $FD_{n,1}$ does not.

Consider a two dimensional random vector on the unit square, where the density sits entirely upon three lines $L_1 = \{(x_1, 0), 0 < x_1 \leq 1\}$, $L_2 = \{(0, x_2), 0 \leq x_2 \leq 1\}$, $L_{12} = \{(x_1, x_2), 0 < x_1 = x_2 \leq 1\}$. We use the (one-dimensional) Lebesgue measure on each of the three line segments and put rescaled uniform densities on L_1 and L_2 , and a rescaled density of g_1 on the diagonal. Hence our density is

$$h_1(x_1, x_2) = \begin{cases} \frac{1}{c_1} & (x_1, x_2) \in L_1 \cup L_2, \\ \frac{1}{c_1} g_1\left(\frac{r}{\sqrt{2}}\right) & (x_1, x_2) \in L_{12}, \\ 0 & \text{otherwise,} \end{cases}$$

where $r = \sqrt{x_1^2 + x_2^2}$, $c_1 = 2 + \sqrt{2}$. It is easy to see that $FD_{2,2}(\{2\}) = 1$, as the number of exceedances is at least 2, but can not exceed 2. However,

$$\frac{\mathbb{P}(N_{s,2} = 1)}{\mathbb{P}(N_{s,2} \geq 2)} = \frac{2\mathbb{P}(X_1 > s, X_2 = 0)}{\mathbb{P}(X_1 = X_2 \geq s)} = \sqrt{2} \cdot \frac{1-s}{1-G_1(s)},$$

which, according to (2.2), does not converge as $s \nearrow$. Therefore, even though $FD_{2,2}$ exists, $FD_{2,1}$ does not.

We now construct another counter-example to show that despite the existence of $FD_{3,1}$, $FD_{3,2}$ does not exist. To this end, let the joint density of (X_1, X_2, X_3) lie only on the following lines: $L_1 = \{(x_1, 0, 0), 0 < x_1 \leq 1\}$, $L_2 = \{(0, x_2, 0), 0 < x_2 \leq 1\}$, $L_3 = \{(0, 0, x_3), 0 < x_3 \leq 1\}$, $L_{12} = \{(x_1, x_2, 0), 0 < x_1 = x_2 \leq 1\}$, $L_{23} = \{(0, x_2, x_3), 0 < x_2 = x_3 \leq 1\}$, $L_{13} = \{(x_1, 0, x_3), 0 < x_1 = x_3 \leq 1\}$, $L_{123} = \{(x_1, x_2, x_3), 0 \leq x_1 = x_2 = x_3 \leq 1\}$, equipped with the (one-dimensional) Lebesgue measure on these lines. We define a new distribution function

$$G_2(z) = \begin{cases} 0 & z < 0, \\ 1 - (1-z)(1-G_1(z)) & z \in [0, 1], \\ 1 & \text{otherwise,} \end{cases}$$

and denote its density by $g_2(z)$. We then set up our joint density of (X_1, X_2, X_3) as

$$h_2(x_1, x_2, x_3) = \begin{cases} \frac{1}{c_2} & (x_1, x_2, x_3) \in L_1 \cup L_2 \cup L_3, \\ \frac{1}{c_2} (\sqrt{2} - r) & (x_1, x_2, x_3) \in L_{12} \cup L_{23} \cup L_{13}, \\ \frac{1}{c_2} g_2\left(\frac{r}{\sqrt{3}}\right) & (x_1, x_2, x_3) \in L_{123}, \\ 0 & \text{otherwise,} \end{cases}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $c_2 = 6 + \sqrt{3}$. Now,

$$\mathbb{P}(N_{s,3} = 1 | N_{s,3} \geq 1) = \frac{3}{\sqrt{3}(1-G_1(s)) + 3(1-s) + 3} \rightarrow 1 \text{ as } s \nearrow,$$

so $FD_{3,1}$ exists. On the other hand, using (2.2), one can show

$$\mathbb{P}(N_{s,3} = 2 | N_{s,3} \geq 2) = \frac{\sqrt{3}}{\frac{1-G_1(s)}{1-s} + \sqrt{3}},$$

does not converge as $s \nearrow$ and hence $FD_{3,2}$ does not exist.

3 Stationary Sequences

In the previous section we considered the fragility distribution in the case when the number of random variables was fixed and finite. We now study properties of $\lim_{n \rightarrow \infty} FD_{n,m}$ for stationary sequences $\{X_i, i \geq 1\}$ with distribution F_X satisfying

$$\lim_{s \nearrow} \frac{1 - F_X(s)}{1 - F_X(s^-)} = 1. \quad (3.1)$$

Let $\mathcal{N}_{u_n,n}(B) = \sum_{i=1}^n \mathbf{1}_{\{\frac{i}{n} \in B, X_i > u_n\}}$ for any Borel $B \subset [0, 1]$, where $\{u_n\}$ is a sequence of constants approaching x_F . Hence $\mathcal{N}_{u_n,n}$ is the time-scaled point process of exceedances and it serves as an instrument for using point process theory to obtain limiting properties in extreme value theory, see for example, Leadbetter et al. (1983), chapter 5.

We say a random variable \mathcal{C} has a compound Poisson distribution $\text{CP}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$, if $\mathcal{C} \stackrel{d}{=} \sum_{i=1}^{\infty} iX_i$, where X_i follows Poisson distribution with mean λ_i , denoted by $\text{Pn}(\lambda_i)$, and the X_i 's are independent. If we write $\lambda = \sum_{i=1}^{\infty} \lambda_i$ and define $\pi_i = \frac{\lambda_i}{\lambda}$, $i \geq 1$, then \mathcal{C} can also be represented as the sum of a $\text{Pn}(\lambda)$ number of i.i.d. random variables with cluster distribution $\boldsymbol{\pi} := \{\pi_i\}$. It has been established since Hsing et al. (1988) that under some mild conditions, the limiting distribution of $N_{u_n,n} = \mathcal{N}_{u_n,n}([0, 1])$ is necessarily compound Poisson. This observation relates the study of extreme value theory to the estimates of the accuracy of compound Poisson approximation for $\mathcal{L}(N_{u_n,n})$, which can be found in, e.g., [3, 5, 11, 12, 18, 19, 20, 21, 23, 24].

By taking appropriate percentiles of the underlying distribution, we can find a normalising sequence $\{u_n^{(\tau)}\}$ such that for any $0 < \tau < \infty$,

$$n(1 - F(u_n^{(\tau)})) \rightarrow \tau \quad \text{as } n \rightarrow \infty.$$

The existence of such sequences $\{u_n^{(\tau)}\}$ is guaranteed by the condition (3.1) (see Leadbetter et al. (1983), Theorem 1.7.13). In this section, assume that we are working with such sequences $u_n^{(\tau)}$ and condition (3.1) holds.

Theorem 3.1. Suppose that $FD_{n,m}$ exists for sufficiently large n and for any $0 < \tau \leq \tau_0$ and $k \geq m$, $\mathbb{P}(N_{u_n^{(\tau)},n}^{(\tau)} = k | N_{u_n^{(\tau)},n}^{(\tau)} \geq m)$ converges uniformly in n to $\mathbb{P}(N^\tau = k | N^\tau \geq m)$, where N^τ is a compound Poisson random variable with rate $\theta\tau$, $\theta > 0$, and cluster distribution $\boldsymbol{\pi}$. Then for any $A \subset \mathbb{Z}_m := \{m, m+1, \dots\}$,

$$\lim_{n \rightarrow \infty} FD_{n,m}(A) = \frac{\boldsymbol{\pi}^{*\mathcal{I}_m}(A)}{\boldsymbol{\pi}^{*\mathcal{I}_m}(\mathbb{Z}_m)},$$

where $\boldsymbol{\pi}^{*j}$ is the convolution of $\boldsymbol{\pi}$ j times and $\mathcal{I}_m = \min\{i : \boldsymbol{\pi}^{*i}(\mathbb{Z}_m) > 0\}$. In particular, we have $\lim_{n \rightarrow \infty} FD_{n,1} = \boldsymbol{\pi}$.

Proof. Applying Theorem 7.11 in Rudin (1976), one can see that the uniform convergence allows the exchange of limits, giving

$$\begin{aligned} \lim_{n \rightarrow \infty} FD_{n,m}(A) &= \lim_{n \rightarrow \infty} \lim_{s \nearrow \infty} \mathbb{P}(N_{s,n} \in A | N_{s,n} \geq m) \\ &= \lim_{n \rightarrow \infty} \lim_{\tau \rightarrow 0} \mathbb{P}(N_{u_n^{(\tau)},n}^{(\tau)} \in A | N_{u_n^{(\tau)},n}^{(\tau)} \geq m) \\ &= \lim_{\tau \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{P}(N_{u_n^{(\tau)},n}^{(\tau)} \in A | N_{u_n^{(\tau)},n}^{(\tau)} \geq m). \end{aligned} \quad (3.2)$$

Since N^τ follows compound Poisson distribution with rate $\theta\tau$ and compounding distribution $\boldsymbol{\pi}$, we can write

$$N^\tau \stackrel{d}{=} \sum_{i=1}^{P_\tau} \xi_i,$$

where P_τ is a Poisson random variable with mean $\theta\tau$, ξ_1, ξ_2, \dots are i.i.d. with distribution $\boldsymbol{\pi}$ and independent of P_τ . Using the law of total probability, by conditioning on P_τ , we obtain from (3.2) that for $A \subset \mathbb{Z}_m$,

$$\begin{aligned} \lim_{n \rightarrow \infty} FD_{n,m}(A) &= \lim_{\tau \rightarrow 0} \mathbb{P}(N^\tau \in A | N^\tau \geq m) \\ &= \lim_{\tau \rightarrow 0} \frac{\boldsymbol{\pi}^{*\mathcal{I}_m}(A) \frac{e^{-\theta\tau}(\theta\tau)^{\mathcal{I}_m}}{\mathcal{I}_m!} + o(\tau^{\mathcal{I}_m})}{\boldsymbol{\pi}^{*\mathcal{I}_m}(\mathbb{Z}_m) \frac{e^{-\theta\tau}(\theta\tau)^{\mathcal{I}_m}}{\mathcal{I}_m!} + o(\tau^{\mathcal{I}_m})} = \frac{\boldsymbol{\pi}^{*\mathcal{I}_m}(A)}{\boldsymbol{\pi}^{*\mathcal{I}_m}(\mathbb{Z}_m)}. \end{aligned}$$

Finally, for $m = 1$, since $\boldsymbol{\pi}(\mathbb{Z}_1) = 1$, then $\mathcal{I}_1 = 1$ and the conclusion follows. \square

In applications, we often face situations where n is fixed and u_n is chosen in such a way that the number of exceedances is within an acceptable range. That is, regardless how large the data set we have, we don't have sufficient information to obtain the exact fragility distribution. Hence, for practical purposes, our interest should be focused on suitable approximations of fragility distributions and their associated error estimates. On the other hand, although it has been established that $\mathcal{L}(N_{s,n})$ can be reasonably approximated by a compound Poisson distribution ([3, 5, 18, 19, 20, 21, 24]), the probability $\mathbb{P}(N_{s,n} \geq m)$ is usually small and the error estimates are often larger than the actual probabilities. For this reason, we can not rely on the estimates of the errors of $\mathbb{P}(N_{s,n} \in \cdot)$ and $\mathbb{P}(N_{s,n} \geq m)$ to work out the approximation errors of $\mathbb{P}(N_{s,n} \in \cdot | N_{s,n} \geq m)$ and it is necessary to study the estimates of $\mathbb{P}(N_{s,n} \in \cdot | N_{s,n} \geq m)$ directly.

4 Conditional compound Poisson Approximation

In the previous section we have seen that the limit fragility distribution can be evaluated by the conditional compound Poisson limit. In this section we will focus on estimating errors of conditional compound Poisson approximation via Stein's method.

For any random variable X , we write $X^{(m)}$ as a random variable having the distribution $\mathcal{L}(X | X \geq m)$, where m is a non-negative integer. For convenience, we define $\text{CP}^{(m)}(\boldsymbol{\lambda}) := \mathcal{L}(\mathcal{C}^{(m)})$, where $\mathcal{C} \sim \text{CP}(\boldsymbol{\lambda})$. The following lemma can be directly verified.

Lemma 4.1. For a non-negative integer m , $W \sim \text{CP}^{(m)}(\boldsymbol{\lambda})$ if and only if for all bounded functions g_m on \mathbb{Z}_m ,

$$\mathbb{E} \left[\sum_{j=1}^{\infty} j \lambda_j g_m(W + j) - W g_m(W) \mathbf{1}_{W > m} \right] = 0. \quad (4.1)$$

Let $\mathcal{B}_m g_m(i) := \sum_{j=1}^{\infty} j \lambda_j g_m(i + j) - i g_m(i) \mathbf{1}_{i > m}$. Our interest is to assess the difference between two distributions Q_1 and Q_2 on \mathbb{Z}_m so we define the total

variation distance as

$$d_{TV}(Q_1, Q_2) := \sup_{f \in \mathcal{F}_m} \left| \int f dQ_1 - \int f dQ_2 \right|,$$

where $\mathcal{F}_m := \{\mathbf{1}_A : A \subset \mathbb{Z}_m\}$. We write Stein's equation as

$$\mathcal{B}_m g_m(i) = f(i) - \text{CP}^{(m)}(\boldsymbol{\lambda})\{f\}, \quad f \in \mathcal{F}_m \quad (4.2)$$

where $\text{CP}^{(m)}(\boldsymbol{\lambda})\{f\} := \mathbb{E}f(\mathcal{C}^{(m)})$ with $\mathcal{C}^{(m)} \sim \text{CP}^{(m)}(\boldsymbol{\lambda})$. Using the same argument as in Theorem 1 in Barbour, Chen & Loh (1992), one can prove that the equation (4.2) has a solution $g_{m,f}$ defined on \mathbb{Z}_m and the solution is unique except at $i = m$.

For a function h on \mathbb{Z}_m , we write $\Delta h(\cdot) = h(\cdot + 1) - h(\cdot)$ and $\|h\|_m = \sup_{w \in \mathbb{Z}_m} |h(w + 1)|$. To apply Stein's method, bounds for

$$G_{m,1} = \sup_{f \in \mathcal{F}_m} \|g_{m,f}\|_m, \quad G_{m,2} = \sup_{f \in \mathcal{F}_m} \|\Delta g_{m,f}\|_m \quad (4.3)$$

are needed. However, as was demonstrated in Barbour, Chen & Loh (1992) and Barbour & Utev (1998, 1999), the estimates of Stein's factors for general compound Poisson approximation are unsatisfactory and useful estimates are only available for special cases. Consequently, we will deal with two special cases: (1) $i\lambda_i$ is decreasing in i and (2) conditional negative binomial approximation.

4.1 Case 1: $i\lambda_i$ is monotone decreasing in i

Theorem 4.2. If $i\lambda_i$ is a decreasing function of i , then both $G_{m,1}$ and $G_{m,2}$ are decreasing in m .

Proof. By setting $g_{m,f}(i) = \Delta h_{m,f}(i - 1)$ (see Barbour (1998) and Barbour, Chen & Loh (1992)), if we assume $i\lambda_i$ is decreasing in i , then the form of Stein's identity in (4.1) naturally leads to a generator interpretation with generator

$$\mathcal{A}_m h_{m,f}(i) := \sum_{j=1}^{\infty} [(j\lambda_j - (j+1)\lambda_{j+1}) (h_{m,f}(i+j) - h_{m,f}(i))]$$

$$+ i(h_{m,f}(i-1) - h_{m,f}(i))\mathbf{1}_{i>m}, \quad i \in \mathbb{Z}_m.$$

Furthermore, it can be verified that the solution for $h_{m,f}(i)$ to Stein equation

$$\mathcal{A}_m h_{m,f}(i) = \mathcal{B}_m g_{m,f}(i) = f(i) - \text{CP}^{(m)}(\boldsymbol{\lambda})\{f\},$$

is

$$h_{m,f}(i) = - \int_0^\infty \left(\mathbb{E}f(Z_i^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m)}) \right) dt,$$

where $Z_i^{(m)}$ is a birth-death process with generator \mathcal{A} and initial value $Z_i^{(m)}(0) = i$. Using the strong Markov property of the birth-death process, we obtain

$$\begin{aligned} h_{m,f}(i+1) &= - \int_0^\infty \left(\mathbb{E}f(Z_{i+1}^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m)}) \right) dt \\ &= -\mathbb{E} \int_0^{\tau_{i+1,i}^{(m)}} \left(f(Z_{i+1}^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m)}) \right) dt + h_{m,f}(i), \end{aligned} \quad (4.4)$$

where $\tau_{i+1,i}^{(m)} = \inf\{t : Z_{i+1}^{(m)}(t) = i\}$. One can replace $f \in \mathcal{F}_m$ with $\mathbf{1}_{\mathbb{Z}_m} - f$ to show that

$$G_{m,1} = \sup_{f \in \mathcal{F}_m} \sup_{i \in \mathbb{Z}_m} g_{m,f}(i+1) = - \inf_{f \in \mathcal{F}_m} \inf_{i \in \mathbb{Z}_m} g_{m,f}(i+1),$$

hence, we can assume that $h_{m,f}(i+1) - h_{m,f}(i) \leq 0$. Now,

$$\begin{aligned} h_{m,f}(i+1) - h_{m,f}(i) &= -\mathbb{E} \int_0^{\tau_{i+1,i}^{(m)}} \left(f(Z_{i+1}^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m)}) \right) dt \\ &= -\mathbb{E} \int_0^{\tau_{i+1,i}^{(m)}} \left(f(Z_{i+1}^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m-1)}) \right) dt \\ &\quad + [\mathbb{E}f(\mathcal{C}^{(m)}) - \mathbb{E}f(\mathcal{C}^{(m-1)})] \mathbb{E}\tau_{i+1,i}^{(m)} \\ &= h_{m-1,f}(i+1) - h_{m-1,f}(i) \\ &\quad + [\mathbb{E}f(\mathcal{C}^{(m)}) - \mathbb{E}f(\mathcal{C}^{(m-1)})] \mathbb{E}\tau_{i+1,i}^{(m)}, \end{aligned} \quad (4.5)$$

where the last equality is because $(Z_{i+1}^{(m)}(\cdot)\mathbf{1}_{\cdot < \tau_{i+1,i}^{(m)}}, \tau_{i+1,i}^{(m)}) \stackrel{d}{=} (Z_{i+1}^{(m-1)}(\cdot)\mathbf{1}_{\cdot < \tau_{i+1,i}^{(m-1)}}, \tau_{i+1,i}^{(m-1)})$ for $i \geq m$. Noting that $f \in \mathcal{F}_m$, we have

$$\mathbb{E}f(\mathcal{C}^{(m)}) - \mathbb{E}f(\mathcal{C}^{(m-1)}) = \frac{\mathbb{E}f(\mathcal{C})}{\mathbb{P}(\mathcal{C} \geq m)} - \frac{\mathbb{E}f(\mathcal{C})}{\mathbb{P}(\mathcal{C} \geq m-1)} \geq 0. \quad (4.6)$$

This, together with (4.5), implies $h_{m-1,f}(i+1) - h_{m-1,f}(i) \leq h_{m,f}(i+1) - h_{m,f}(i)$. Therefore,

$$\begin{aligned} -G_{m,1} &= \inf_{f \in \mathcal{F}_m} \inf_{i \in \mathbb{Z}_m} (h_{m,f}(i+1) - h_{m,f}(i)) \\ &\geq \inf_{f \in \mathcal{F}_m} \inf_{i \in \mathbb{Z}_m} (h_{m-1,f}(i+1) - h_{m-1,f}(i)) \\ &\geq \inf_{f \in \mathcal{F}_{m-1}} \inf_{i \in \mathbb{Z}_{m-1}} (h_{m-1,f}(i+1) - h_{m-1,f}(i)) = -G_{m-1,1}. \end{aligned}$$

The proof of the monotonicity of $G_{m,2}$ is similar. Again replacing $f \in \mathcal{F}_m$ with $1_{\mathbb{Z}_m} - f$ if necessary, we can prove $G_{m,2} = \sup_{f \in \mathcal{F}_m} \sup_{i \in \mathbb{Z}_m} \Delta g_{m,f}(i+1) = -\inf_{f \in \mathcal{F}_m} \inf_{i \in \mathbb{Z}_m} \Delta g_{m,f}(i+1)$. Hence we can assume $\Delta_2 h_{m,f}(i) := h_{m,f}(i+2) - 2h_{m,f}(i+1) + h_{m,f}(i) \geq 0$. Arguing in the same way as for (4.5), we have

$$\begin{aligned} \Delta_2 h_{m,f}(i) &= -\mathbb{E} \int_0^{\tau_{i+2,i+1}^{(m)}} (f(Z_{i+2}^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m)}))dt \\ &\quad + \mathbb{E} \int_0^{\tau_{i+1,i}^{(m)}} (f(Z_{i+1}^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m)}))dt \\ &= -\mathbb{E} \int_0^{\tau_{i+2,i+1}^{(m)}} (f(Z_{i+2}^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m-1)}))dt \\ &\quad + \mathbb{E} \int_0^{\tau_{i+1,i}^{(m)}} (f(Z_{i+1}^{(m)}(t)) - \mathbb{E}f(\mathcal{C}^{(m-1)}))dt \\ &\quad + (\mathbb{E}f(\mathcal{C}^{(m)}) - \mathbb{E}f(\mathcal{C}^{(m-1)}))(\mathbb{E}\tau_{i+2,i+1}^{(m)} - \mathbb{E}\tau_{i+1,i}^{(m)}) \\ &= \Delta_2 h_{m-1,f}(i) + (\mathbb{E}f(\mathcal{C}^{(m)}) - \mathbb{E}f(\mathcal{C}^{(m-1)}))(\mathbb{E}\tau_{i+2,i+1}^{(m)} - \mathbb{E}\tau_{i+1,i}^{(m)}). \end{aligned}$$

It can be shown via a coupling that $\mathbb{E}\tau_{i+2,i+1}^{(m)} - \mathbb{E}\tau_{i+1,i}^{(m)} \leq 0$. Hence, it follows from (4.6) that $\Delta_2 h_{m,f}(i) \leq \Delta_2 h_{m-1,f}(i)$, which ensures

$$\begin{aligned} G_{m,2} &= \sup_{f \in \mathcal{F}_m} \sup_{i \in \mathbb{Z}_m} \Delta_2 h_{m,f}(i+1) \leq \sup_{f \in \mathcal{F}_m} \sup_{i \in \mathbb{Z}_m} \Delta_2 h_{m-1,f}(i+1) \\ &\leq \sup_{f \in \mathcal{F}_{m-1}} \sup_{i \in \mathbb{Z}_{m-1}} \Delta_2 h_{m-1,f}(i+1) = G_{m-1,2}. \end{aligned}$$

□

As a direct consequence of the above theorem, we can use the bounds for compound Poisson approximation given in Barbour, Chen & Loh (1992) to give crude estimates of $G_{m,1}$ and $G_{m,2}$.

Corollary 4.3. If $j\lambda_j$ is a decreasing function of j , then for any m ,

$$G_{m,1} \leq \begin{cases} 1 & \text{if } \lambda_1 - 2\lambda_2 \leq 1 \\ (1/\sqrt{\lambda_1 - 2\lambda_2}) [2 - (1/\sqrt{\lambda_1 - 2\lambda_2})] & \text{if } \lambda_1 - 2\lambda_2 > 1 \end{cases},$$

$$G_{m,2} \leq 1 \wedge \frac{1}{\lambda_1 - 2\lambda_2} \left[\frac{1}{4(\lambda_1 - 2\lambda_2)} + \log^+ 2(\lambda_1 - 2\lambda_2) \right].$$

We will see in the next subsection that the bounds can be improved for conditional Poisson and conditional negative binomial approximations. However, in the general case, it remains a challenging open problem to find the optimal estimates of $G_{m,1}$ and $G_{m,2}$.

4.2 Case 2: Conditional Negative Binomial Approximation

We use the following parameterisation of the negative binomial distribution. Let a random variable Z have negative binomial distribution with parameters r and p , denoted by $\text{NB}(r, p)$, if it has mass function

$$\mathbb{P}(Z = k) = \frac{\Gamma(r + k)}{\Gamma(r)k!} (1 - p)^r p^k, \quad k = 0, 1, \dots; r > 0, 0 < p < 1.$$

The negative binomial distribution can be viewed as a compound Poisson distribution with cluster distribution following a logarithmic distribution (see Johnson et al. (2005), p. 223). It can also be considered as the stationary distribution of a birth-death process with linear birth rate and unit per capita death rate (Phillips (1996)). The latter consideration leads us to the following observation.

Lemma 4.4. For a non-negative integer m , $W \sim \text{NB}^{(m)}(r, p)$ if and only if for all bounded functions g_m on \mathbb{Z}_m ,

$$\mathbb{E} [p(r + W)g_m(W + 1) - Wg_m(W)\mathbf{1}_{W > m}] = 0. \quad (4.7)$$

Again we solve for the function $g_m := g_{m,f}$ that satisfies Stein's equation

$$\mathbb{B}_m g_m(i) := p(r+i)g_m(i+1) - ig_m(i)\mathbf{1}_{i>m} = f(i) - \text{NB}^{(m)}(r,p)\{f\}, \quad (4.8)$$

where $\text{NB}^{(m)}(r,p)\{f\} := \mathbb{E}(f(Z^{(m)}))$ with $Z^{(m)} \sim \text{NB}^{(m)}(r,p)$, and calculate bounds for $G_{m,1}$ and $G_{m,2}$.

Theorem 4.5. For conditional negative binomial approximation, both $G_{m,1}$ and $G_{m,2}$ are decreasing in m .

We omit the proof of this theorem as it is essentially the same as the proof of Theorem 4.2.

Using the bound for $G_{0,1}$ in Brown and Phillips (1999), and the previous theorem we achieve the following.

Corollary 4.6. For conditional negative binomial approximation, the solution to Stein's equation (4.8) satisfies

$$G_{m,1} \leq \frac{1}{1-p} \wedge \frac{1.75}{\sqrt{rp(1-p)}}.$$

Unlike in the compound Poisson case where we use the unconditional bound of $G_{0,2}$ to bound $G_{m,2}$, we can obtain a tight bound of $G_{m,2}$ for negative binomial approximation.

Theorem 4.7. For conditional negative binomial approximation, the solution to Stein's equation (4.8) satisfies

$$G_{m,2} = \frac{\mathbb{P}(Z > m)}{p(r+m)\mathbb{P}(Z \geq m)}, \quad (4.9)$$

where $Z \sim \text{NB}(r,p)$.

Proof. Similarly to the previous subsection, by setting $g_m(i) = h_m(i) - h_m(i-1)$, the form of Stein's identity in (4.7) leads to a generator interpretation with generator

$$\mathbb{A}_m h_m(i) := p(r+i)(h_m(i+1) - h_m(i)) + i(h_m(i-1) - h_m(i))\mathbf{1}_{i>m}.$$

It is a routine exercise to show that the stationary distribution of a process with this generator is $\text{NB}^{(m)}(r, p)$. Noting that our process satisfies (C4) in Brown & Xia (2001), we obtain from Theorem 2.10 of Brown & Xia (2001) that

$$\sup_{f \in \mathcal{F}_m} |\Delta g_{m,f}(i)| = \frac{1 - \pi_m^{(m)} - \dots - \pi_i^{(m)}}{p(r+i)} + \frac{\pi_m^{(m)} + \dots + \pi_{i-1}^{(m)}}{i},$$

where $\pi_j^{(m)} = \mathbb{P}(Z^{(m)} = j)$. Using the balance equations of stationary processes, we also have that $(j+1)\pi_{j+1}^{(m)} = p(r+j)\pi_j^{(m)}$. Therefore, rearranging the above equation gives

$$\begin{aligned} \sup_{f \in \mathcal{F}_m} |\Delta g_{m,f}(i)| &= \frac{1 - \pi_m^{(m)}}{p(r+m)} + (\pi_{i+1}^{(m)} + \pi_{i+2}^{(m)} + \dots) \frac{m-i}{p(r+i)(r+m)} \\ &\quad + \sum_{k=m}^{i-1} \left(\frac{\pi_k^{(m)}}{i} - \frac{\pi_{k+1}^{(m)}}{p(r+m)} \right) \\ &\leq \frac{1 - \pi_m^{(m)}}{p(r+m)}, \end{aligned} \tag{4.10}$$

since $m \leq i$ and for $m \leq k \leq i-1$,

$$\frac{\pi_k^{(m)}}{i} - \frac{\pi_{k+1}^{(m)}}{p(r+m)} = \pi_k^{(m)} \left(\frac{1}{i} - \frac{r+k}{(k+1)(r+m)} \right) \leq \frac{\pi_k^{(m)}(m-k)}{(k+1)(r+m)} \leq 0.$$

Direct verification ensures that the inequality of (4.10) becomes an equality when $i = m$. \square

For conditional Poisson approximation, it is natural to set Stein's equation

$$C_m g_m(i) := \lambda g_m(i+1) - i g_m(i) \mathbf{1}_{i>m} = f(i) - \text{Pn}^{(m)}(\lambda) \{f\}, \tag{4.11}$$

where $\text{Pn}^{(m)}(\lambda) \{f\} := \mathbb{E}(f(P^{(m)}))$ with $P^{(m)} \sim \text{Pn}^{(m)}(\lambda)$. We define $G_{m,1}$ and $G_{m,2}$ as in (4.3).

Corollary 4.8. For conditional Poisson approximation, the solution to Stein's equation (4.11) satisfies

$$G_{m,1} \leq 1 \wedge \sqrt{\frac{2}{\lambda e}},$$

$$G_{m,2} = \frac{\mathbb{P}(P > m)}{\lambda \mathbb{P}(P \geq m)},$$

where $P \sim \text{Pn}(\lambda)$.

Proof. The bound of $G_{m,1}$ follows from Remark 3.4 of Barbour & Brown (1992) and Theorem 4.5. For the second estimate, since $\text{NB}(r, p)$ approaches $\text{Poisson}(\lambda)$ when $r \rightarrow \infty$ and $rp \rightarrow \lambda$, therefore the bound follows by taking the limit in Theorem 4.7. \square

Remark. The advantages of dealing with the conditional approximation directly can be seen as follows. For $\text{Pn}^{(1)}(\lambda)$ approximation, $G_{1,2} = \frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda(1-e^{-\lambda})}$. In the case where λ is small, this tends towards $\frac{1}{2}$. If one were to use Stein's method for Poisson approximation, then $G_{0,2} = \frac{1-e^{-\lambda}}{\lambda}$, which has a limit of 1 for small λ . Therefore, using the conditional approximation appropriately, one can typically reduce the error bound by a factor of $\frac{1}{2}$ when λ is small.

It is worthwhile to point out that the representation (4.7) enables us to relate an estimate of conditional negative binomial approximation to that of the corresponding negative binomial approximation. This property does not seem to be shared by the general conditional compound Poisson approximation.

Lemma 4.9. If a nonnegative integer valued random variable W can be approximated by $\text{NB}(r, p)$ and it can be shown that

$$|\mathbb{E}B_0g_0(W)| \leq \epsilon_1\|g_0\|_0 + \epsilon_2\|\Delta g_0\|_0,$$

for all functions g_0 on \mathbb{Z}_0 for which $\|g_0\|_0$ and $\|\Delta g_0\|_0$ are finite, and ϵ_1, ϵ_2 are positive, then $W^{(m)}$ can be approximated by $\text{NB}^{(m)}(r, p)$ with

$$d_{TV}(\mathcal{L}(W^{(m)}), \text{NB}^{(m)}(r, p)) \leq \frac{1}{\mathbb{P}(W \geq m)} \{\epsilon_1 G_{m,1} + \epsilon_2 G_{m,2}\}.$$

Proof. For each $f \in \mathcal{F}_m$, we define $L_f(w) = \begin{cases} g_{m,f}(w), & \text{for } w > m, \\ 0, & \text{for } w \leq m, \end{cases}$ then it follows from (4.8) that

$$\left| \mathbb{E}f(W^{(m)}) - \text{NB}^{(m)}(r, p)\{f\} \right|$$

$$\begin{aligned}
&= \left| \mathbb{E} \left(p(r + W^{(m)}) g_{m,f}(W^{(m)} + 1) - W^{(m)} g_{m,f}(W^{(m)}) \mathbf{1}_{W^{(m)} > m} \right) \right| \\
&= \frac{|\mathbb{E} B_0 L_f(W)|}{\mathbb{P}(W \geq m)} \leq \frac{\epsilon_1 \|L_f\|_0 + \epsilon_2 \|\Delta L_f\|_0}{\mathbb{P}(W \geq m)}.
\end{aligned}$$

However, $\|\Delta L_f\|_0 = G_{m,2} \vee \sup_{f \in \mathcal{F}_m} |\Delta L_f(m)|$, therefore it remains to show that $\sup_{f \in \mathcal{F}_m} |\Delta L_f(m)| = \sup_{f \in \mathcal{F}_m} |g_{m,f}(m+1)| \leq G_{m,2}$. To this end, similar to the derivation of (4.4) we use the strong Markov property to obtain

$$\begin{aligned}
g_{m,f}(m+1) &= \mathbb{E} \int_0^{\tau_{m,m+1}^{(m)}} [f(Z_m^{(m)}(t)) - \text{NB}^{(m)}(r, p)(f)] dt \\
&= \frac{1}{p(r+m)} \left(f(m) - \text{NB}^{(m)}(r, p)(f) \right),
\end{aligned}$$

which ensures $|g_{m,f}(m+1)| \leq \frac{1 - \mathbb{P}(Z^{(m)}=m)}{p(r+m)} = G_{m,2}$. \square

Remark. Some care is needed when we apply Lemma 4.9 because of the small probability $\mathbb{P}(W \geq m)$ in the denominator. However, in reality, the most interesting case is for $m = 1$ and we will show that the error bounds for approximations with $m = 1$ are usually small.

4.3 Applications

In our first example, we consider the exceedances of a sequence of independent but not necessarily identically distributed random variables.

Example 4.10. Let X_i , $i \in \{1, \dots, n\}$ be independent random variables with $\mathbb{P}(X_i > s) = p_i$ and $N_{s,n}$ be the number of exceedances above s , then

$$d_{TV}(\mathcal{L}(N_{s,n}^{(1)}), \text{Pn}^{(1)}(\lambda)) \leq \frac{\frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda(1-e^{-\lambda})} \sum_{i=1}^n p_i^2}{1 - \prod_{i=1}^n (1 - p_i)}, \quad (4.12)$$

where $\lambda = \sum_{i=1}^n p_i$.

In the case where n is fixed, $p_i = p$ for all i and p is small, then the bound is asymptotically $\frac{p}{2}$.

Proof. The claim easily follows from Lemma 4.9, Corollary 4.8 and equation (1.23) from Barbour, Holst & Janson (1992). \square

Example 4.11. Let $\{Y_i, 1 \leq i \leq n\}$ be i.i.d. random variables with $p = \mathbb{P}(Y_i > s)$ and $X_i = Y_i \wedge Y_{i+1}, 1 \leq i \leq n$, where $Y_{n+1} := Y_1$. With $b = \frac{2p-3p^2}{1+2p-3p^2}$ and $a = (1-b)np^2$, the number $N_{s,n}$ of exceedances of $\{X_i, 1 \leq i \leq n\}$ above s satisfies

$$d_{TV}(\mathcal{L}(N_{s,n}^{(1)}), \text{NB}^{(1)}(a/b, b)) \leq \frac{32.2p}{\sqrt{(n-1)(1-p)^3}} \cdot \frac{aG_{1,2}}{\mathbb{P}(N_{s,n} \geq 1)}, \quad (4.13)$$

with $G_{1,2}$ defined in (4.9). When n is fixed and p is small, the upper bound is asymptotically $\frac{16.1p}{\sqrt{(n-1)(1-p)^3}}$.

Proof. Using Lemma 4.9, one can exploit the proof of Theorem 4.2 in Brown & Xia (2001) word for word, with Stein's factors replaced by their conditional equivalents, to get the first claim. For the asymptotic result, Theorem 8.G in Barbour, Holst & Janson (1992) gives

$$\mathbb{P}(N_{s,n} \geq 1) \geq 1 - e^{-\lambda(1-p)} - (5p^2 - 4p^3),$$

which, after some elementary expansion, yields $\frac{aG_{1,2}}{\mathbb{P}(N_{s,n} \geq 1)} \asymp \frac{1}{2}$. \square

Lemma 4.9 essentially states that if a random variable W of interest can be well approximated by a negative binomial random variable, then its conditional distribution can also be well approximated by the conditional negative binomial distribution. The following example shows that the converse is not true. In other words, the conditional negative binomial approximation may be appropriate even if the unconditional approximation is poor.

Consider a sequence of random variables X_1, \dots, X_n which are conditionally independent given a parameter random variable Θ , where Θ takes values 0 or 1 with distribution $\mathbb{P}(\Theta = 1) = 1 - \mathbb{P}(\Theta = 0) = q$. When the parameter $\Theta = 0$, exceedances will not happen, while $\Theta = 1$ is the phase where exceedances may happen. Such phenomena are very common in quality control, seismology and finance. In quality control (Lambert (1992)), if manufacturing equipment is well maintained, it will not fail, but when the equipment

is wearing out due to insufficient maintenance, it has a positive probability to fail during its operation. In seismology (Ellsworth & Beroza (1995)), earthquakes are strongly linked to a distinctive seismic nucleation phase. In finance (Yalamova & McKelvey (2011)), the “herding behavior” is often linked to different phases of the financial market with some dying off and some leading to crashes. These phenomena can not be modelled by a Poisson distribution as the errors of approximation are too large to justify a Poisson approximation. For this reason, a zero-inflated Poisson is a more suitable choice (Lambert (1992)).

Example 4.12. With the setup in the preceding paragraph, let $p_1 = \mathbb{P}(X_1 > s | \Theta = 1)$ and $\lambda = np_1$, then the number $N_{s,n}$ of exceedances satisfies

$$d_{TV}(\mathcal{L}(N_{s,n}^{(1)}), \text{Pn}^{(1)}(\lambda)) \leq \frac{p_1(1 - e^{-\lambda} - \lambda e^{-\lambda})}{1 - (1 - p_1)^n}$$

and the bound is asymptotically $\frac{1}{2}p_1$ for fixed n and small p_1 .

Proof. For convenience, we write $W = N_{s,n}$ and we wish to approximate $W^{(1)}$ with $\text{Pn}^{(1)}(\lambda)$. To this end, we observe from (4.11) that

$$\begin{aligned} & \left| \mathbb{E}f(W^{(1)}) - \text{Pn}^{(1)}(\lambda)\{f\} \right| = \left| \mathbb{E}[\text{C}_1 g_{1,f}(W^{(1)})] \right| \\ &= \left| \frac{\mathbb{E}[\text{C}_1 g_{1,f}(W) \mathbf{1}_{W \geq 1}]}{\mathbb{P}(W \geq 1)} \right| = \left| \frac{\mathbb{E}[\text{C}_1 g_{1,f}(W) \mathbf{1}_{W \geq 1} | \Theta = 1]}{\mathbb{P}(W \geq 1 | \Theta = 1)} \right| \\ &\leq \frac{G_{1,2} n p_1^2}{1 - (1 - p_1)^n}, \end{aligned}$$

where the inequality is derived as in the proof of Example 4.10. \square

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